

# EXPLICIT FORMULAS INVOLVING $q$ -EULER NUMBERS AND POLYNOMIALS

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**ABSTRACT.** In this paper, we deal with  $q$ -Euler numbers and  $q$ -Bernoulli numbers. We derive some interesting relations for  $q$ -Euler numbers and polynomials by using their generating function and derivative operator. Also, we show between the  $q$ -Euler numbers and  $q$ -Bernoulli numbers via the  $p$ -adic  $q$ -integral in the  $p$ -adic integer ring.

## 1. PRELIMINARIES

Imagine that  $p$  be a fixed odd prime number. Throughout this paper we use the following notations, by  $\mathbb{Z}_p$  denotes the ring of  $p$ -adic rational integers,  $\mathbb{Q}$  denotes the field of rational numbers,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers, and  $\mathbb{C}_p$  denotes the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ .

The  $p$ -adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

In this paper we assume  $|q - 1|_p < 1$  as an indeterminate.

$[x]_q$  is a  $q$ -extension of  $x$  which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q},$$

we note that  $\lim_{q \rightarrow 1} [x]_q = x$  (see[1-12]).

We say that  $f$  is a uniformly differentiable function at a point  $a \in \mathbb{Z}_p$ , if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

has a limit  $f'(a)$  as  $(x, y) \rightarrow (a, a)$  and denote this by  $f \in UD(\mathbb{Z}_p)$ .

Let  $UD(\mathbb{Z}_p)$  be the set of uniformly differentiable function on  $\mathbb{Z}_p$ . For  $f \in UD(\mathbb{Z}_p)$ , let us start with the expressions

$$\frac{1}{[p^N]_q} \sum_{0 \leq \xi < p^N} f(\xi) q^\xi = \sum_{0 \leq \xi < p^N} f(\xi) \mu_q(\xi + p^N \mathbb{Z}_p),$$

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represents  $p$ -adic  $q$ -analogue of Riemann sums for  $f$ . The integral of  $f$  on  $\mathbb{Z}_p$  will be defined as the limit ( $N \rightarrow \infty$ ) of these sums, when it exists. The  $p$ -adic  $q$ -integral of function  $f \in UD(\mathbb{Z}_p)$  is defined by T. Kim

$$(1.1) \quad I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{\xi=0}^{p^N-1} f(\xi) q^\xi$$

The bosonic integral is considered as a bosonic limit  $q \rightarrow 1$ ,  $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$ . Similarly, the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  is introduced by T. Kim as follows:

$$(1.2) \quad I_{-q}(f) = \lim_{q \rightarrow -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi)$$

(for more details, see [9-12]).

In [6], the  $q$ -Euler polynomials with weight 0 are introduced as

$$(1.3) \quad \tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y)$$

From (1.3), we have

$$\tilde{E}_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} x^l \tilde{E}_{n-l,q}$$

where  $\tilde{E}_{n,q}(0) = \tilde{E}_{n,q}$  are called  $q$ -Euler numbers with weight 0. Then,  $q$ -Euler numbers are defined as

$$q \left( \tilde{E}_q + 1 \right)^n + \tilde{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention about replacing  $\left( \tilde{E}_q \right)^n$  by  $\tilde{E}_{n,q}$  is used.

Similarly, the  $q$ -Bernoulli polynomials and numbers with weight 0 are defined, respectively

$$\begin{aligned} \tilde{B}_{n,q}(x) &= \lim_{n \rightarrow \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x+y)^n q^y \\ &= \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y) \end{aligned}$$

and

$$\tilde{B}_{n,q} = \int_{\mathbb{Z}_p} y^n d\mu_q(y)$$

(for more informations, see [4]).

We, by using Kim's et al. method in [2], will investigate some interesting identities on the  $q$ -Euler numbers and polynomials from their generating function and derivative operator. Consequently, we derive some properties on  $q$ -Euler numbers and polynomials and  $q$ -Bernoulli numbers and polynomials by using  $q$ -Volkenborn integral and fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ .

## 2. ON KIM'S $q$ -EULER NUMBERS AND POLYNOMIALS

Let us consider Kim's  $q$ -Euler polynomials as follows:

$$(2.1) \quad F_x^q = F_x^q(t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}.$$

Here  $x$  is a fixed parameter. Thus, by expression of (2.1), we can readily see the following

$$(2.2) \quad qe^t F_x^q + F_x^q = [2]_q e^{xt}.$$

Last from equality, taking derivative operator  $D$  as  $D = \frac{d}{dt}$  on the both sides of (2.2). Then, we easily see that

$$(2.3) \quad qe^t (D + I)^k F_x^q + D^k F_x^q = [2]_q x^k e^{xt}$$

where  $k \in \mathbb{N}^*$  and  $I$  is identity operator. By multiplying  $e^{-t}$  on both sides of (2.3), we get

$$(2.4) \quad q(D + I)^k F_x^q + e^{-t} D^k F_x^q = [2]_q x^k e^{(x-1)t}$$

Let us take derivative operator  $D^m$  ( $m \in \mathbb{N}$ ) on both sides of (2.4). Then we get

$$(2.5) \quad qe^t D^m (D + I)^k F_x^q + D^k (D - I)^m F_x^q = [2]_q x^k (x - 1)^m e^{xt}$$

Let  $G[0]$  (not  $G(0)$ ) be the constant term in a Laurent series of  $G(t)$ . Then, from (2.5), we get

$$(2.6) \quad \sum_{j=0}^k \binom{k}{j} (qe^t D^{k+m-j} F_x^q(t)) [0] + \sum_{j=0}^m \binom{m}{j} (-1)^j (D^{k+m-j} F_x^q(t)) [0] = [2]_q x^k (x - 1)^m$$

By (2.1), we see

$$(2.7) \quad (D^N F_x^q(t)) [0] = \tilde{E}_{N,q}(x) \text{ and } (e^t D^N F_x^q(t)) [0] = \tilde{E}_{N,q}(x)$$

By expressions of (2.6) and (2.7), we see that

$$(2.8) \quad \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j,q}(x) = [2]_q x^k (x - 1)^m.$$

From (2.1), we note that

$$(2.9) \quad \frac{d}{dx} (\tilde{E}_{n,q}(x)) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \tilde{E}_{l,q} x^{n-1-l} = n \tilde{E}_{n-1,q}(x)$$

By (2.9), we easily see,

$$(2.10) \quad \int_0^1 \tilde{E}_{n,q}(x) dx = \frac{\tilde{E}_{n+1,q}(1) - \tilde{E}_{n+1,q}}{n+1} = -\frac{[2]_{q^{-1}}}{n+1} \tilde{E}_{n+1,q}$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$\begin{aligned}
 (2.11) \quad & -[2]_{q^{-1}} \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\widetilde{E}_{k+m-j+1,q}}{k+m-j+1} \\
 & = [2]_q (-1)^m B(k+1, m+1) \\
 & = [2]_q (-1)^m \frac{\Gamma(k+1) \Gamma(m+1)}{\Gamma(k+m+2)}
 \end{aligned}$$

where  $B(m, n)$  is beta function which is defined by

$$\begin{aligned}
 B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\
 &= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0 \text{ and } n > 0.
 \end{aligned}$$

As a result, we obtain the following theorem

**Theorem 1.** *For  $n \in \mathbb{N}$ , we have*

$$\begin{aligned}
 & \sum_{j=1}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\widetilde{E}_{k+m-j+1,q}}{k+m-j+1} \\
 & = q \frac{(-1)^{m+1}}{(k+m+1) \binom{k+m}{k}} - [2]_q \frac{\widetilde{E}_{k+m+1,q}}{k+m+1}.
 \end{aligned}$$

Substituting  $m = k+1$  into Theorem 1, we readily get

$$\begin{aligned}
 & \sum_{j=1}^{k+1} \left[ q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \frac{\widetilde{E}_{2k+2-j,q}}{2k+2-j} \\
 & = q \frac{(-1)^k}{(2k+2) \binom{2k+1}{k}} - [2]_q \frac{\widetilde{E}_{2k+2,q}}{2k+2}.
 \end{aligned}$$

By (2.1), it follows that

$$\begin{aligned}
 & \sum_{j=0}^{\max\{k,m\}} (k+m-j) \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \widetilde{E}_{k+m-j-1,q}(x) \\
 & = [2]_q x^{k-1} (x-1)^{m-1} ((k+m)x - k).
 \end{aligned}$$

Let  $m = k$  in (2.1), we see that

$$\sum_{j=0}^k \left[ q \binom{k}{j} + (-1)^j \binom{k}{j} \right] \widetilde{E}_{2k-j,q}(x) = [2]_q x^k (x-1)^k$$

Last from equality, we discover the following

$$\begin{aligned}
 (2.12) \quad & [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \widetilde{E}_{2k-2j,q}(x) + (q-1) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \widetilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x-1)^k.
 \end{aligned}$$

Here  $[\cdot]$  is Gauss' symbol. Then, taking integral from 0 to 1 both sides of last equality, we get

$$\begin{aligned}
& -[2]_{q^{-1}} [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + [2]_{q^{-1}} (1-q) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\
& = [2]_q (-1)^k B(k+1, k+1) \\
& = \frac{[2]_q (-1)^k}{(2k+1) \binom{2k}{k}}.
\end{aligned}$$

Consequently, we derive the following theorem

**Theorem 2.** *The following identity*

$$\begin{aligned}
(2.13) \quad & [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + (q-1) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\
& = \frac{q(-1)^{k+1}}{(2k+1) \binom{2k}{k}}.
\end{aligned}$$

is true.

In view of (2.1) and (2.12), we discover the following applications:

$$\begin{aligned}
(2.14) = & \sum_{j=0}^{k+1} \left[ q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \tilde{E}_{2k+1-j,q}(x) \\
& = [2]_q \tilde{E}_{2k+1,q}(x) + \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \left[ q \binom{k}{2j} + \binom{k}{2j} + \binom{k}{2j-1} \right] \tilde{E}_{2k+1-2j,q}(x) \\
& \quad + \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor} \left[ q \binom{k}{2j+1} - \binom{k}{2j+1} - \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x) \\
& = - \left[ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \tilde{E}_{2k-2j+1,q}(x) \right] \\
& \quad + [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
& \quad + (q-1) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \tilde{E}_{2k-2j+1,q}(x)
\end{aligned}$$

By expressions (2.12) and (2.14), we have the following Theorem

**Theorem 3.** For  $k \in \mathbb{N}$ , we have

$$\begin{aligned}
 (2.15) \quad & [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\
 & + (q-1) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left[ \tilde{E}_{2k-2j,q}(x) + \frac{1}{1+q} \tilde{E}_{2k-2j+1,q}(x) \right] \\
 & = x^k (x-1)^k ([2]_q x - q)
 \end{aligned}$$

### 3. $p$ -adic integral on $\mathbb{Z}_p$ associated with Kim's $q$ -Euler polynomials

In this section, we consider Kim's  $q$ -Euler polynomials by means of  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ . Now we start with the following assertion.

Let  $m, k \in \mathbb{N}$ , Then by (2.8),

$$\begin{aligned}
 I_1 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}
 \end{aligned}$$

On the other hand, right hand side of (2.8),

$$\begin{aligned}
 I_2 &= \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
 &= \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}
 \end{aligned}$$

Equating  $I_1$  and  $I_2$ , we get the following theorem

**Theorem 4.** For  $m, k \in \mathbb{N}$ , we have

$$\begin{aligned}
 & \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q} \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}.
 \end{aligned}$$

Let us take fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  left hand side of (2.15), we get

$$\begin{aligned}
 I_3 &= \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]_q x - q) d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_{-q}(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q}
 \end{aligned}$$

In other word, we consider right hand side of (2.15) as follows:

$$\begin{aligned}
I_4 &= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left[ \begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \end{aligned} \right] \\
&= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left[ \begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{aligned} \right]
\end{aligned}$$

Equating  $I_3$  and  $I_4$ , we get the following theorem

**Theorem 5.** For  $k \in \mathbb{N}$ , we have

$$\begin{aligned}
&\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left[ [2]_q \tilde{E}_{k+l+1,q} - q \tilde{E}_{k+l,q} \right] \\
&= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left\{ \begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{aligned} \right\}
\end{aligned}$$

Now, we consider (2.8) and (2.1) by means of  $q$ -Volkenborn integral. Then, by (2.8), we see

$$\begin{aligned}
&[2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_q(x) \\
&= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x) \\
&= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q}
\end{aligned}$$

On the other hand,

$$\begin{aligned} & \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ = & \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q} \end{aligned}$$

Therefore, we get the following theorem

**Theorem 6.** *For  $m, k \in \mathbb{N}$ , we have*

$$\begin{aligned} & [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q} \\ = & \sum_{j=0}^{\max\{k,m\}} \left[ q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q} \end{aligned}$$

By using fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  left hand side of (2.15), we get

$$\begin{aligned} I_5 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]x-q) d\mu_q(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,q} \end{aligned}$$

Also, we consider right hand side of (2.15) as follows:

$$\begin{aligned} I_6 &= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &+ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &+ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left[ \frac{(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)}{+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x)} \right] \\ &= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\ &+ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\ &+ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left[ \frac{(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q}}{+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1,q} \tilde{B}_{l,q}} \right] \end{aligned}$$

Equating  $I_5$  and  $I_6$ , we get the following Corollary

**Corollary 1.** For  $k \in \mathbb{N}$ , we get

$$\begin{aligned}
& \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left[ [2]_q \tilde{B}_{k+l+1,q} - q \tilde{B}_{k+l,q} \right] \\
&= [2]_q \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
&+ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
&+ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j+1} \left\{ \begin{aligned} & (q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q} \\ & + \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1,q} \tilde{B}_{l,q} \end{aligned} \right\}
\end{aligned}$$

#### REFERENCES

- [1] Araci, S., Erdal, D., and Seo, J.-J., A study on the fermionic  $p$ -adic  $q$ -integral representation on  $\mathbb{Z}_p$  associated with weighted  $q$ -Bernstein and  $q$ -Genocchi polynomials, Abstract and Applied Analysis, Volume **2011**, Article ID 649248, 10 pages.
- [2] T. Kim, B. Lee, S. H. Lee, S.-H. Rim, Identities for the Bernoulli and Euler numbers and polynomials, Accepted in Ars Combinatoria.
- [3] Kim, D., Kim, T., Lee, S.-H., Dolgy, D.-V., and Rim, S.-H., Some new identities on the Bernoulli numbers and polynomials, Discrete Dynamics in Nature and Society, Volume **2011**, Article ID 856132, 11 pages.
- [4] Kim, T., Choi, J., and Kim, Y.-H., Some identities on the  $q$ -Bernoulli numbers and polynomials with weight 0, Abstract And Applied Analysis, Volume **2011**, Article ID 361484, 8 pages.
- [5] Kim, T., On a  $q$ -analogue of the  $p$ -adic log gamma functions related integrals, J. Number Theory, **76** (1999) no. 2, 320-329.
- [6] Kim, T., and Choi, J., On the  $q$ -Euler numbers and polynomials with weight 0, Abstract and Applied Analysis, Volume **2012**, ID 795304, 7 pages, doi:10.1155/2012/795304.
- [7] Kim, T., On the  $q$ -extension of Euler and Genocchi numbers, J. Math. Anal. Appl. **326** (2007) 1458-1465.
- [8] Kim, T., On the weighted  $q$ -Bernoulli numbers and polynomials, Advanced Studies in Contemporary Mathematics **21**(2011), no.2, p. 207-215, <http://arxiv.org/abs/1011.5305>.
- [9] Kim, T.,  $q$ -Volkenborn integration, Russ. J. Math. phys. **9**(2002), 288-299.
- [10] Kim, T.,  $q$ -Euler numbers and polynomials associated with  $p$ -adic  $q$ -integrals, J. Nonlinear Math. Phys., **14** (2007), no. 1, 15-27.
- [11] Kim, T., New approach to  $q$ -Euler polynomials of higher order, Russ. J. Math. Phys., **17** (2010), no. 2, 218-225.
- [12] Kim, T., Some identities on the  $q$ -Euler polynomials of higher order and  $q$ -Stirling numbers by the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ , Russ. J. Math. Phys., **16** (2009), no.4, 484-491.

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